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Deriving the Dyer-Roeder Equation from the Geodesic Deviation Equation via the Newman-Penrose Null Tetrad

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The Dyer-Roeder equation is an equation used for calculating distances to astronomical objects. The Dyer-Roeder equation approximates the universe to have a uniform density in all directions, i.e. that the universe is homogeneous and isotropic [Foster et al (1995)]. The metric used to derive the equation assumes there are no clumps of matter in the space-time, which makes the equation for distance simple enough to derive. The assumption that the universe has a uniform density limits the scope of objects to which we can calculate the distance. If light from an astronomical object on its way to earth, passes through the gravitational field of a clump of matter then we cannot calculate the distance to that object using Dyer-Roeder equation.

Attempts to find a solvable expression for angular diameter distance using a metric that allows for clumps of matter have historically been unfruitful. We think this failure is a direct result of using basis vectors that are best suited to deal with flat space-times rather than curved space-times. In

this paper we attempt to show that the N-P null-tetrad of basis vectors is better suited for the curvature of space-time associated with clumpy cosmologies.

The N-P formalism for General Relativity is useful in dealing with motion of light-bundles, or a propagating pencil of light rays, in a curved space-time. It allows us to deal with problems arising from the curvature of space-time due to local variations in matter density by introducing the null tetrad of basis vectors. In the phrase “null-tetrad”, the word “null” means light-like and “tetrad” means a set of four. The reason we need four vectors is due to the fact that we are working in a four dimensional space-time which requires four independent bases vector to span the whole space. Since astronomers study astronomical objects by observing light emitted from these objects, it makes sense that we would use the null-tetrad of basis vectors when deriving an equation for the distance to these objects. Before this can be accomplished we should be able to show that the N-P null-tetrad can produce the equation for angular diameter distance for a flat FLRW. Once this is shown to be the case we can calculate angular diameter distance for the perturbed FLRW.

This derivation of the Dyer-Roeder equation is a first step in obtaining an equation for angular diameter distance in a perturbed FLRW metric using the N-P tetrad. We start our derivation with a discussion about the null tetrad. We then use the flat FLRW metric to calculate the N-P components needed to solve the geodesic deviation equation for angular diameter distance. Finally we make the

appropriate substitutions to get the Dyer-Roeder equation.

Mathematical Background The General FLRW Metric

The most general form of the FLRW, as discussed in section 1, allows for global curvature. The metric can be expressed to account for all three possible global curvatures in a homogeneous and isotropic universe.

$$ds^2 = dt^2 - a^2(t) \begin{cases} \text{Closed:} & d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \\ \text{Flat:} & dx^2 + dy^2 + dz^2 \\ \text{Open:} & d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \end{cases} \quad (1)$$

The metric is expressed for the three possible geometries. The possible geometries arise from the EFE equation(2, [Carroll, S. (2004), p. ~332].

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \quad (2)$$

In the EFE the metric is expressed as a second rank tensor. T_{ab} is the stress tensor which codes the distribution of pressure, matter, and energy. R_{ab} and R are the Ricci tensor and scalar respectively (see section 1).

Using

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b) \quad (3)$$

,where u is a velocity four vector and ρ is the mass density of our dust model. In the flat case, the case in which we are interested, the metric can be expressed in terms of the scale factor ($a(t)$), as:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \quad (4)$$

Null-Tetrad

For our derivation we will use the N-P formalism. This formalism makes use of a tetrad, or a set of

four, basis vectors associated with a light ray. The null-tetrad is:

$$\mathcal{L}_i^a = \{l^a, n^a, m^a, \bar{m}^a\}, \quad (5)$$

Where l^a, n^a, m^a, \bar{m}^a is a set of four basis vectors. In our tetrad, l^a is tangent to the light ray, n^a is perpendicular to l^a in the metric space/plane. m^a and \bar{m}^a are complex axial vectors for a cross section which slices an ellipsoid shaped bundle of light-rays (see figure 1). The proposed null-tetrad in terms of the parameter ξ and the cosmological scale factor $a(t)$, are:

$$l^a = \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})} \langle (-1-\xi\bar{\xi}), \frac{1}{a(t)}(\bar{\xi}+\xi), \frac{i}{a(t)}(\bar{\xi}-\xi), \frac{1}{a(t)}(-1+\xi\bar{\xi}) \rangle, \quad (6)$$

$$n^a = \frac{a(t)}{\sqrt{2}(1+\xi\bar{\xi})} \langle (-1-\xi\bar{\xi}), \frac{-1}{a(t)}(\bar{\xi}+\xi), \frac{i}{a(t)}(\xi-\bar{\xi}), \frac{1}{a(t)}(-1+\xi\bar{\xi}) \rangle, \quad (7)$$

$$m^a = \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})} \langle 0, (1-\bar{\xi}^2), -i(\bar{\xi}^2+1), 2\bar{\xi} \rangle, \quad (8)$$

$$\bar{m}^a = \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})} \langle 0, (1-\xi^2), i(\xi^2+1), 2\xi \rangle. \quad (9)$$

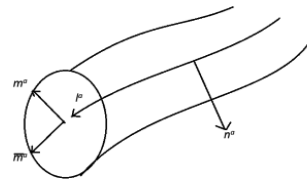


Figure 1: Shows an ellipsoid light-bundle traveling along the l^a direction. The tetrad are represented by the four arrows in the figure

This is the general expression for the N-P null-tetrad in Cartesian, x, y, z , coordinates. In equations (2.2), ξ & $\bar{\xi}$ are stereographic projections onto a complex plane. They map all points on a semi-

sphere onto a flat complex plane (see figure 2).

$$\xi = \cot\left(\frac{\theta}{2}\right) e^{i\phi}, \quad (10)$$

where $0 < \theta < \pi$ and $0 \leq \phi \leq 2\pi$.

In this derivation we can make some assumptions about the space-time and the tetrad to make them simpler. Since Dyer-Roeder assumes a flat homogeneous and isotropic universe, in our derivation, we assume the same thing. If we are looking at an object that is directly overhead, the light from that object will not have components in the x-y plane (this amounts to choosing an origin for a coordinate system). The only direction along which the light ray should progress is the z-direction, using standard Cartesian coordinates (see figure 3).

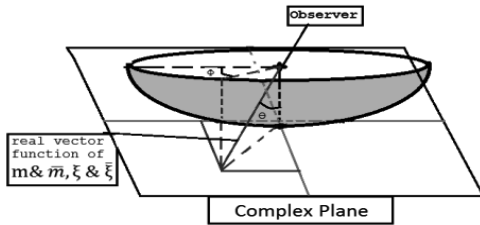


Figure 2: Shows an observer at the center of a semi-sphere of light. The semi-sphere of light ray-vectors $v = v(\xi, \bar{\xi}, m, \bar{m})$ is mapped onto a complex plane.

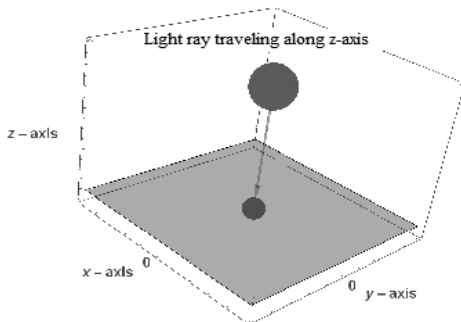


Figure 3: Shows a light emitting object directly overhead. The light ray only travels in the z-direction.

The vector l^a is tangent to the light-ray, which means that $l^a = \frac{1}{\sqrt{2}} \langle \dot{t}, \dot{x}, \dot{y}, \dot{z} \rangle$ and since we

have no movement in the x-y plane, $\dot{x} = 0$ and $\dot{y} = 0$. In terms of our stereographic coordinates,

ξ and $\bar{\xi}$, the angle θ is measured from the center of a sphere with respect to the point at which the semi-sphere touches the plane (see figure 2). This strategic choice of coordinates makes the task of calculating distance easy by making $\xi = \bar{\xi} = 0$.

Our null tetrad now has spatial components in only the z direction, and the time component remains unchanged.

$$l^a = \frac{1}{a(t)\sqrt{2}} \langle -1, 0, 0, \frac{-1}{a(t)} \rangle, \quad (11)$$

And since $\xi = \bar{\xi} = 0$, m^a becomes:

$$m^a = \frac{1}{a(t)\sqrt{2}} \langle 0, 1, -i, 0 \rangle. \quad (12)$$

Now that we have l^a and m^a we solve for the N-P components needed to solve the geodesic deviation equation. To find an expression for angular diameter “distance”, or the equivalent of Dyer-Roeder equation, for a perturbed FLRW metric, we will need to use the tetrad in equations (2.2).

Note: n^a, m^a , and \bar{m}^a all have components that are NOT tangent to the light-ray. We only make that argument for l^a and use $\xi = \bar{\xi} = 0$ to get the rest of the tetrad in the derivation that follows.

Angular Diameter Distance in a Flat, Homogeneous, and Isotropic Universe

Angular Diameter Distance is a way to talk about the distance to faraway objects of known size.

Suppose there is a sphere of radius (l). When viewed by an observer from some distance (d), it subtends an angle θ as measured by the observer (see figure 4). If the angle is small enough, then the diameter is approximately equal to the arc length. The arc length is a product of the radius and the angle of the arc. Here the radius is the Angular Diameter-Distance (D_A).

$$D_A = \frac{l}{\theta} \quad (13)$$

Astronomers use the angular diameter distance to estimate the distance to an object. This is done by measuring the angle subtended by an object of known size. If the angular diameter distance of the object is known, then its size can be estimated using the same relation.

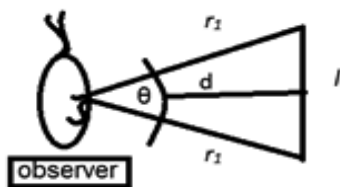


Figure 4: The angular diameter distance is related to the Luminosity Distance (d_L) by redshift (z_R) and comoving transverse distance (d_M) [Ryden, B. (2003)]:

$$d_L = d_M (1 + z_R) \quad (14)$$

$$D_A = \frac{d_M}{1 + z_R}. \quad (15)$$

The comoving transverse distance (d_M) is defined in terms of the proper transverse velocity (u) as measured using redshift. The velocity is calculated

using $u = \frac{c}{H_0} \frac{(1 - z_R)^2}{(1 + z_R)^2}$ and the angular velocity

($\dot{\theta}$) as measured relative to an observer [Carroll, S. (2004), pp.~344-349].

$$d_M = \frac{u}{\dot{\theta}} \quad (16)$$

Due to the expansion of the universe with time, astronomers must consider the effects the expansion has on measurement and calculation. One of these effects is that in an expanding universe, a fixed, non-expanding, coordinate system will give different coordinates for objects which are at rest otherwise.

To deal with this difficulty we make use of a comoving coordinate system. This is a system of coordinates that expands at the same rate as that of the universe, allowing objects that move due to expansion only to keep the same coordinates. The actual distance is then obtained via a coordinate transformation. The comoving distance is then the separation distance between the source of the light and the observer in this expanding coordinate system. The comoving distance (transverse) between any two objects in this system is the separation distance between the two points. This comoving distance is not the actual distance one would travel if one wanted to get to the object in question. To get the actual distance we need to include the cosmic scaling factor which is a parameter with a magnitude that varies with time.

Note: this is only true for a Flat, homogenous, and Isotropic universe.

Derivation

Calculating the N-P Components for the Flat FLRW Metric Using the Tetrad

We start our derivation of the Dyer-Roeder equation from the geodesic deviation equation by

choosing a cosmology or a metric. The metric we use is the FLRW metric for a flat cosmology as discussed in section (2.1). This is an expanding cosmology that has a uniform matter density in every direction, and it is flat everywhere. We can express the flat FLRW metric in two important ways:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \quad (17)$$

$$g_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{bmatrix}, \quad (18)$$

The N-P components, needed for the geodesics deviation equation, are:

$$\Phi_{00} = -\frac{1}{2} R_{ab} l^a l^b \quad (19)$$

$$\Psi_0 = -C_{abcd} l^a m^b l^c m^d \quad (20)$$

Where R_{ab} is the Ricci tensor and C_{abcd} is the Weyl tensor. These tensors encode the curvature of the space in question and they are calculated by contracting the Riemann tensor R_{bcd}^a . The Ricci tensor is the symmetric part of the Riemann tensor and it is expressed as:

$$R_{ab} = R_{acb}^c, \quad (21)$$

where the Riemann tensor is contracted along the repeated index. The Weyl tensor is the curvature tensor “with all of its contractions removed”, it is the anti-symmetric part of the Riemann tensor [Carroll, S. (2004)]. For a four dimensional manifold

the Weyl tensor is:

$$C_{abcd} = R_{abcd} - g_{a[c} R_{d]b} - g_{b[c} R_{d]a} + \frac{1}{3} g_{a[c} R_{d]b}. \quad (22)$$

The Riemann tensor $R_{abcd} = g_{ae} R_{bcd}^e$ is defined in terms of the Levi-Civita connection Γ_{bc}^a as:

$$R_{bcd}^a = \partial_c \Gamma_{db}^a - \partial_d \Gamma_{cb}^a + \Gamma_{ce}^a \Gamma_{db}^e - \Gamma_{cb}^e \Gamma_{de}^a. \quad (23)$$

We can calculate the Levi-Civita Connection Γ_{bc}^a from the metric tensor, equation (1). The connection is defined to be:

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}). \quad (24)$$

We used the xAct package in Mathematica to find the components of the connection, Ricci, and Weyl tensors. The surviving connection terms are [Wald, R. M. (1984), p.~ 97]:

$$\Gamma_{ii}^0 = a\dot{a} \quad (25)$$

$$\Gamma_{0i}^i = \Gamma_{i0}^i = a\dot{a} \quad (26)$$

for $i=1,2,3$. The Weyl tensor vanishes in a flat FLRW, or $C_{abcd} = 0$ in equation (20), and we are only left with the Ricci terms. The Ricci tensor has only the following components:

$$R_{00} = \frac{-3\ddot{a}}{a} \quad (27)$$

and

$$R_{ii} = a\ddot{a} + 2\dot{a}^2, \quad (28)$$

for $i=1, 2, 3$. Now that we have these components we can calculate Φ_{00} by substituting equations (1) into equation (19), or:

$$\Phi_{00} = \frac{-1}{2} R_a b^a l^b = \frac{-1}{2} [R_0 l^0 l^0 + R_1 l^1 l^1 + R_2 l^2 l^2 + R_3 l^3 l^3]$$

$$\Phi_{00} = \frac{-1}{2} \left[\frac{-3\ddot{a}}{a} (l^0)^2 + (a\ddot{a} + 2\dot{a}^2)(l^1)^2 + (a\ddot{a} + 2\dot{a}^2)(l^2)^2 + (a\ddot{a} + 2\dot{a}^2)(l^3)^2 \right]$$

Substituting in our tetrad:

$$\Phi_{00} = \frac{-1}{2} \left[\frac{-3\ddot{a}}{a} \left(\frac{1}{a^2 2} \right) + (a\ddot{a} + 2\dot{a}^2) (0)^2 + (a\ddot{a} + 2\dot{a}^2) (0)^2 + (a\ddot{a} + 2\dot{a}^2) \left(\frac{1}{a^2 2} \right) \right]$$

Then by removing the zero terms, distribution, and expansion we get,

$$\Phi_{00} = \frac{-1}{2} \left[\left(\frac{-3\ddot{a}}{2a^3} \right) + \frac{a\ddot{a}}{2a^4} + \frac{2\dot{a}^2}{2a^4} \right] = \frac{-1}{2a^2} \left[\left(\frac{-3\ddot{a}}{2a} \right) + \frac{\ddot{a}}{2a} + \frac{\dot{a}^2}{a^2} \right] = \frac{-1}{2a^2} \left[\left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a} \right]$$

We can write our final expression so that it is ready for use in the derivation of Dyer-Roeder as:

$$\Phi_{00} = \frac{1}{2a^2} \left[\left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a} \right]. \quad (29)$$

Now that we have Φ_{00} we are ready for the final derivation of the Dyer-Roeder equation using the N-P null-tetrad.

Geodesic Deviation and Angular Diameter Distance

The Dyer-Roeder equation is an equation for angular diameter distance. Angular diameter distance, as discussed in section 2.3, is the length of an object divided by the angle subtended by the object according to some observer in a flat FLRW cosmology. For our derivation of angular diameter distance we rely on the geodesic deviation equation. The geodesic deviation equation will supply the diameter of the object, or (θ) , from section 2.3. Geodesic deviation refers to the behavior of rays of light as they travel through some space and how that behavior deviates from a linear behavior. Two

light rays traveling in a flat space will diverge linearly or not at all (figure 5). In order for the light rays to diverge non-linearly they need to be accelerated. This acceleration, in a clumpy cosmology, is provided by the curvature of the space-time. We measure this acceleration by taking the second derivative of the displacement vector between the rays of a light bundle with respect to time. If this derivative is a constant the space is flat, if it is not a constant the space is curved (see figure 5).

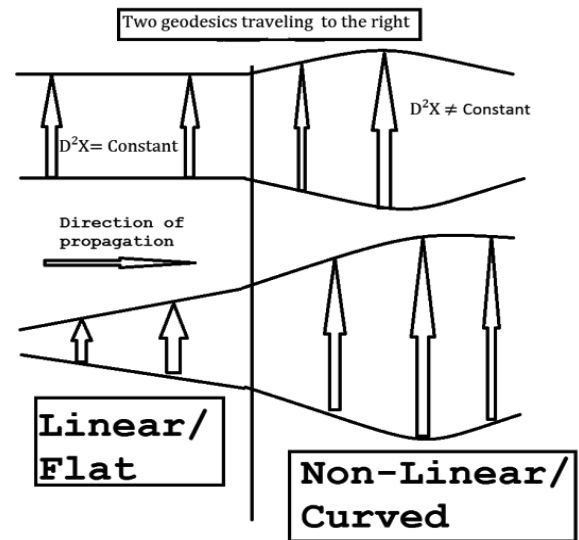


Figure 5: Shows two sets of two light rays (geodesics) traveling through a flat space on the left and a curved space on the right. The geodesic deviation vector X is represented by the arrows and changed linearly in the flat space and non-linearly in the curved space. The second order differential operator D^2 measures the extent to which the geodesics are accelerating.

In the N-P formalism the second derivative is replaced by a second order differential operator (D^2). We apply this operator to a set of two complex vectors ζ & η and their complex conjugates. Together with the complex vectors m & \bar{m} from our null tetrad, we can calculate the

real vector q for the displacement of light rays in a light bundle.

$$q = \zeta \bar{m}^a + \bar{\eta} m^a \quad (30)$$

In the N-P formalism the geodesic deviation vectors are collected in the matrix X .

$$X = \begin{bmatrix} \zeta & \eta \\ \bar{\eta} & \bar{\zeta} \end{bmatrix} \quad (31)$$

The differential operator (D) is the operator used in the N-P formalism [Kling Campbell(2008)]. By applying (D)two times to the deviation vectors we are in effect measuring the distortion of an image as viewed by an observer due to some acceleration caused by the curvature of the space-time. The differential operator (D) is given by a the change along the light ray, (h):

$$D = l^a \frac{\partial}{\partial x^a}. \quad (32)$$

We will show that dividing the deviation vectors X by the angle subtended in an observer's sphere of light will yield the equation for angular diameter distance as described in section 2.3,

$$D_A = \frac{l}{\theta} \rightarrow \frac{X}{\theta} \quad (33)$$

which is the Dyer-Roeder equation. We can then solve the geodesic deviation equation for the deviation vectors X . In the N-P formalism is

$$D^2 X = QX \quad (34)$$

where, (Q) is the matrix which codes for the effects of the space time on the deviation vectors,

$$Q = \begin{bmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{bmatrix}. \quad (35)$$

As discussed in section (1)the Weyl tensor vanishes and the Ricci component is given by equation (19).

Deriving the Dyer-Roeder Equation from the Geodesic Deviation Equation

Our goal is to start with the geodesic deviation equation and to derive the Dyer-Roeder equation. The geodesic deviation equation can be expressed in the N-P formalism as,

$$D^2 X = QX. \quad (36)$$

In this equation Q, X , are given by equations (35) and (31). The matrix product QX in equation (34) is calculated as:

$$QX = \begin{bmatrix} \Phi_{00}\zeta + \Psi_0\bar{\eta} & \Phi_{00}\eta + \Psi_0\bar{\zeta} \\ \bar{\Psi}_0\zeta + \Phi_{00}\bar{\eta} & \bar{\Psi}_0\eta + \Phi_{00}\bar{\zeta} \end{bmatrix}. \quad (37)$$

Since $D^2 X$ can be represented in matrix form as

$$D^2 X = \begin{bmatrix} D^2 \zeta & D^2 \eta \\ D^2 \bar{\eta} & D^2 \bar{\zeta} \end{bmatrix}, \text{ we can write the}$$

following four equations:

$$D^2 \zeta = \Phi_{00}\zeta + \Psi_0\bar{\eta} \quad (38)$$

$$D^2 \eta = \Phi_{00}\eta + \Psi_0 \quad (39)$$

$$D^2 \bar{\eta} = \bar{\Psi}_0\zeta + \Phi_{00}\bar{\eta} \quad (40)$$

$$D^2 \bar{\zeta} = \bar{\Psi}_0\eta + \Phi_{00}\bar{\zeta} \quad (41)$$

In this case D is an operator which acts on the components of the matrix X as follows:[2]

$$D = l^a \frac{\partial}{\partial x^a} = l^0 \frac{\partial}{\partial t} + l^3 \frac{\partial}{\partial z}. \quad (42)$$

Which means $D^2 = l^a \frac{\partial}{\partial x^a} l^b \frac{\partial}{\partial x^b}$. The Dyer-Roeder equation calculates distances in a flat-homogeneous cosmology, or a cosmology

described by the FLRW metric, where Φ_{00} is equation (19) and $\Psi_0 = 0$. Since the Dyer-Roeder equation is written in terms of red-shift distance (z_R) and matter density (Ω_m) we need a change of variable from time (t) in equation (42) to the red-shift distance and matter density. We don't need to worry about the z derivative in equation (42) because neither l_a nor Φ_{00} depend on z and the second terms in equations (38)-(41) vanish. To accomplish a change of variables we utilized the Hubble parameter (Ryden 2003).

$$H \equiv \frac{\dot{a}}{a}, \quad (43)$$

The null vector l^a :

$$l^a = \frac{1}{a\sqrt{2}} \langle -1, 0, 0, \frac{-1}{a} \rangle, \quad (44)$$

And the relationship between red-shift distance and the cosmological scale factor $a(t)$. This is generally

$$\frac{a(t_0)}{a(t)} = 1 + z_R, \text{ where } (t_0) \text{ is the time now, so by}$$

letting $a(t_0)=1$ we can write,

$$\frac{1}{a(t)} = 1 + z_R. \quad (45)$$

By solving equation (45) for z_R and taking the derivative of both sides with respect to time we get,

$$\frac{\partial}{\partial t} = \frac{-\dot{a}}{a^2} \frac{\partial}{\partial z_R}. \quad (46)$$

As discussed, since nothing in equation (17) depends on z , and by substituting equations (43 - 46) into equation (42) we get:

$$D = l^0 \frac{\partial}{\partial t} = -l^0 \frac{\dot{a}}{a^2} \frac{\partial}{\partial z_R} = -\left(-\frac{1}{a\sqrt{2}}\right) \left(\frac{\dot{a}}{a^2} \frac{\partial}{\partial z_R}\right) = \left(\frac{1}{a^2\sqrt{2}}\right) \left(H \frac{\partial}{\partial z_R}\right),$$

or,

$$D = \frac{\sqrt{2}}{2} (1+z)^2 H \frac{\partial}{\partial z_R}. \quad (47)$$

Operating with this differential operator two times can be expressed in terms of equation (43) and equation (45) by taking the appropriate derivatives and simplifying.

$$D^2 = \frac{1}{2} H(1+z_R)^2 \frac{\partial}{\partial z_R} H(1+z_R)^2 \frac{\partial}{\partial z_R} \quad (48)$$

First we deal with the derivatives,

$$\frac{\partial}{\partial z_R} H(1+z_R)^2 \frac{\partial}{\partial z_R} = H(1+z_R)^2 \frac{\partial^2}{\partial^2 z_R} + \left(\frac{\partial}{\partial z_R} H(1+z_R)^2\right) \frac{\partial}{\partial z_R}$$

and,

$$\left(\frac{\partial}{\partial z_R} H(1+z_R)^2\right) \frac{\partial}{\partial z_R} = (1+z_R)^2 \frac{\partial H}{\partial z_R} \frac{\partial}{\partial z_R} + 2(1+z_R)H \frac{\partial}{\partial z_R}.$$

Then by combining the last two equations and simplifying,

$$H(1+z_R)^2 \frac{\partial^2}{\partial^2 z_R} + (1+z_R)^2 \frac{\partial H}{\partial z_R} \frac{\partial}{\partial z_R} + 2(1+z_R)H \frac{\partial}{\partial z_R} = H(1+z_R)^2 \frac{\partial^2}{\partial^2 z_R} + \left((1+z_R)^2 \frac{\partial H}{\partial z_R} + 2(1+z_R)H\right) \frac{\partial}{\partial z_R}$$

.Substituting this result into equation (48) gives,

$$D^2 = \left[\frac{1}{2} H(1+z_R)^2\right] \left[H(1+z_R)^2 \frac{\partial^2}{\partial^2 z_R} + \left((1+z_R)^2 \frac{\partial H}{\partial z_R} + 2(1+z_R)H\right) \frac{\partial}{\partial z_R}\right].$$

By distributing $H(1+z_R)^2$ we can write,

$$D^2 = \frac{1}{2} \left[H^2(1+z_R)^4 \frac{\partial^2}{\partial^2 z_R} + \left((1+z_R)^4 H \frac{\partial H}{\partial z_R} + 2(1+z_R)^3 H^2\right) \frac{\partial}{\partial z_R} \right].$$

We now factor out a $\frac{1}{2}$ and we have D^2 in terms

of redshift distance and the Hubble parameter in a form that will become useful later in our derivation.

$$D^2 = \frac{1}{4} \left[\left[(1+z_R) 2H \frac{\partial H}{\partial z_R} + 4H^2 \right] (1+z_R)^3 \frac{\partial}{\partial z_R} + 2H^2(1+z_R)^4 \frac{\partial^2}{(\partial z_R)^2} \right]. \quad (49)$$

Since in a flat FLRW cosmology $\Psi_0 = 0$, equation (38) becomes $D^2\zeta = \Phi_{00}\zeta$. Dividing both sides by the angle α (see section 2.3), where $D_A = \frac{\zeta}{\alpha}$ and D_A is the angular diameter distance, we get [Ryden, B. (2003)]:

$$D^2 D_A = \Phi_{00} D_A. \quad (50)$$

Then by substituting equations (49) and (29) into equation (50) our expression becomes:

$$\frac{1}{4} \left[\left((1+z_R) 2H \frac{\partial H}{\partial z_R} + 4H^2 \right) (1+z_R)^3 \frac{\partial D_A}{\partial z_R} + 2H^2 (1+z_R)^4 \frac{\partial^2 D_A}{(\partial z_R)^2} \right] = -\frac{1}{2a^2} \left[\left(\frac{\Phi}{a} \right)^2 - \frac{\Phi}{a} \right] D_A, \quad (51)$$

From equation (43) we have:

$$\frac{\partial H}{\partial t} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2. \quad (52)$$

Where we substitute equations (46) and (52) into equation (51) we get,

$$\frac{1}{4} \left[\left((1+z_R) 2H \frac{\partial H}{\partial z_R} + 4H^2 \right) (1+z_R)^3 \frac{\partial D_A}{\partial z_R} + 2H^2 (1+z_R)^4 \frac{\partial^2 D_A}{(\partial z_R)^2} \right] = \frac{-1}{2a^3} \left[\frac{\dot{a}}{a} \frac{\partial H}{\partial z_R} \right] D_A. \quad (53)$$

By combining equations (43) (45) and (53), then setting the expression equal to zero and multiplying both sides by 4 we get:

$$\left[2H^2 (1+z_R)^4 \right] \frac{\partial^2 D_A}{(\partial z_R)^2} + \left[(1+z_R)^4 2H \frac{\partial H}{\partial z_R} + 4(1+z_R)^3 H^2 \right] \frac{\partial D_A}{\partial z_R} + \left[(1+z_R)^3 2H \frac{\partial H}{\partial z_R} \right] D_A = 0. \quad (54)$$

The only variable that does not depend on red-shift distance in equation (54) is the Hubble parameter and its derivatives which depend on t indirectly through $a(t)$. The Hubble parameter can be expressed in terms of redshift distance by the

Hubble constant H_0 . For the most general case, not flat and uniform, can be expressed as:

$$H^2 = H_0^2 \left[\frac{\Omega_m}{a^3} + \frac{1-\Omega_m-\Omega_\Lambda}{a^2} + \Omega_\Lambda \right] = H_0^2 [\Omega_m (1+z_R)^3 + (1-\Omega_m-\Omega_\Lambda)(1+z_R)^2 + \Omega_\Lambda] \quad (55)$$

We are trying to find the Dyer-Roeder equation as it appears in Ehlers(1992). In this version the author assumes that $\Omega_\Lambda = 0$. In our metric we assume that the cosmology is flat, or that $1-\Omega_m-\Omega_\Lambda=0$ [[Carroll,S. (2004)]. This reduces equation (55) to:

$$H^2 = H_0^2 \Omega_m (1+z_R)^3. \quad (56)$$

Taking the derivative of both sides with respect to the red-shift distance we get,

$$\frac{\partial H^2}{\partial z_R} = 2H \frac{\partial H}{\partial z_R} = 3H_0^2 \Omega_m (1+z_R)^2. \quad (57)$$

By substituting equation (56) and (57) into equation (54) we get,

$$\begin{aligned} & \left[2(1+z_R)^4 \right] \left[\Omega_m H_0^2 (1+z_R)^3 \right] \frac{\partial^2 D_A}{(\partial z_R)^2} \\ & + \left[\left((1+z_R)^4 (3H_0^2 \Omega_m (1+z_R)^2) + 4(1+z_R)^3 (\Omega_m H_0^2 (1+z_R)^3) \right) \right] \frac{\partial D_A}{\partial z_R} \\ & + \left[3H_0^2 \Omega_m (1+z_R)^3 (1+z_R)^2 \right] D_A = 0 \end{aligned} \quad (58)$$

Reorganizing we get,

$$\begin{aligned} & \left[2H_0^2 (1+z_R)^7 \Omega_m \right] \frac{\partial^2 D_A}{(\partial z_R)^2} \\ & + \left[\left(3H_0^2 (1+z_R)^6 \Omega_m \right) + \left(4H_0^2 (1+z_R)^6 \Omega_m \right) \right] \frac{\partial D_A}{\partial z_R} \\ & + 3H_0^2 \Omega_m (1+z_R)^5 D_A = 0 \end{aligned} \quad (59)$$

Dividing both sides by $H_0^2(1+z_R)^5$ and combining the coefficients of the first derivative in the differential equation we get,

$$2(1+z_R)^2 \Omega_m \frac{\partial^2 D_A}{(\partial z_R)^2} + 7(1+z_R) \Omega_m \frac{\partial D_A}{\partial z_R} + 3 \Omega_m D_A = 0. \quad (60)$$

I did not divide by Ω_m in the last simplification in order to make it clear that we have forced $\Omega_m = 1$ because we assumed $0 = 1 - \Omega_m - \Omega_\Lambda$ and Ehlers assumes $\Omega_\Lambda = 0$. When we divide both sides of equation (60) by 2 and setting $\Omega_m = 1$ we get:

$$(1+z_R)^2 \frac{\partial^2 D_A}{(\partial z_R)^2} + \frac{7}{2}(1+z_R) \frac{\partial D_A}{\partial z_R} + \frac{3}{2} D_A = 0. \quad (61)$$

This is our final expression for angular diameter distance in a flat FLRW cosmology. This is the same expression that is found in Ehlers', after setting $\Omega = 1 + \frac{kc^2}{(R_0 H_0)^2} = 1$. For flat FLRW $k=0$. The Actual expression for Dyer-Roeder on pg. 137 of Ehlers' book is:

$$(z+1)(\Omega z+1) \frac{\partial^2 D}{dz^2} + \left(\frac{7}{2}\Omega z + \frac{\Omega}{2} + 3\right) \frac{\partial D}{\partial z} + \frac{3}{2}\Omega D = 0. \quad (62)$$

By setting $\Omega=1$ Ehler's expression reduces to:

$$(z+1)(z+1) \frac{\partial^2 D}{dz^2} + \left(\frac{7}{2}z + \frac{1}{2} + 3\right) \frac{\partial D}{\partial z} + \frac{3}{2} D = 0.$$

Finally, by combining like terms and factoring the second term, Ehler's equation matches our expression from equation (61), where $z_R = z$, or:

$$(z+1)^2 \frac{\partial^2 D}{dz^2} + \frac{7}{2}(1+z) \frac{\partial D}{\partial z} + \frac{3}{2} D = 0, \quad (63)$$

This means that by using the N-P formalism and starting from the geodesic deviation equation in a flat FLRW cosmology, we have reproduced the Dyer-Roeder equation.

Conclusion

We have shown that the N-P formalism can give us an equation for angular diameter distance that matches those obtained using traditional coordinate basis. We used the flat FLRW metric and the null tetrad to derive the Dyer-Roeder equation from the geodesic deviation equation. Our derivation is an expression for angular diameter distance in terms of red-shift distance. We were able to confirm that we have the right expression for angular diameter distance by comparing it to the Dyer-Roeder equation in Ehlers(1992). Now that we have shown the N-P tetrad capable of producing the Dyer-Roeder equation for a flat FLRW metric, we think it is possible derive the Dyer-Roeder equation for angular diameter distance in a perturbed FLRW cosmology from the geodesic deviation equation. This derivation also serves as proof that the Dyer-Roeder equation is the geodesic deviation equation.

Next Steps

As discussed in the introduction, our final expression is only valid for objects to which we have a clear line of sight. This restriction is a result of starting with the the unperturbed flat FLRW metric. For objects to which we do not have a clear line-of-sight, we must account for the curvature

produced by the presence of clumps of matter along the line-of-sight. This can be done by perturbing the FLRW metric with gravitational potential and allowing for global curvature. By using the null tetrad we think, it is possible to find useful expressions for angular diameter distance in a perturbed FLRW cosmology. We believe that using the null-tetrad will simplify the mathematics and allow us to solve the the geodesic deviation equation for angular diameter distance. If so, the resulting equation could prove to be a useful tool for astronomers looking at objects through gravitational lenses.

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